

Integer Linear Programming

CSCI 532

If you had the integer hull, Simplex would easily find the optimum. Calculating the integer hull is usually harder than solving the ILP.

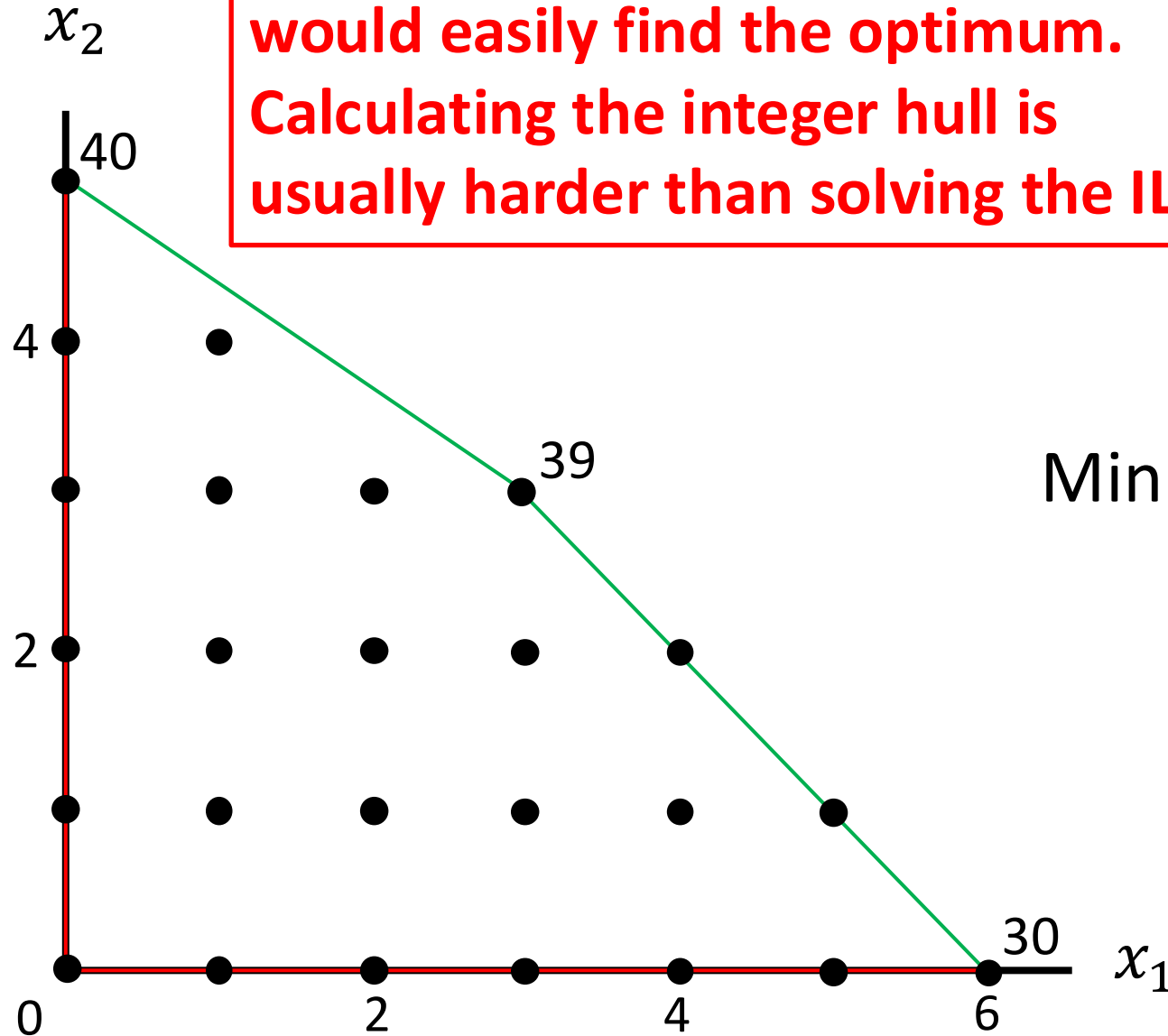
$$x_1, x_2 \in \mathbb{N}$$

Objective: $\max 5x_1 + 8x_2$

Subject to: $x_1 + x_2 \leq 6$

$$5x_1 + 9x_2 \leq 45$$

$$x_1, x_2 \geq 0$$



Minimal convex hull (integer hull):

- Convex.
- local optimum = global optimum.
- $O\left(n^{\lfloor d/2 \rfloor}\right)$ faces,
 $n = \#$ points
 $d = \#$ dimensions

ILP vs LP

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

Optimal:

$$x_1 = 0$$

$$x_2 = 1$$

$$x_3 = 1$$

$$x_4 = 1$$

$$x_5 = 0$$

Objective = 3

$x_i \in [0,1]$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

Optimal:

$$x_1 = 0.5$$

$$x_2 = 0.5$$

$$x_3 = 0.5$$

$$x_4 = 0.5$$

$$x_5 = 0.5$$

Objective = 2.5

Since the LP has **more options** to reduce the objective value, $OPT_{LP} \leq OPT_{ILP}$ (for a minimization problem). If the minimum objective value comes from an integer solution, a plain LP solver (e.g., Simplex) will find it.

Solving ILPs

2. Solve relaxed LP. Suppose the optimal objective is 100 with a fractional solution (e.g. some $x_i = 3.24$).

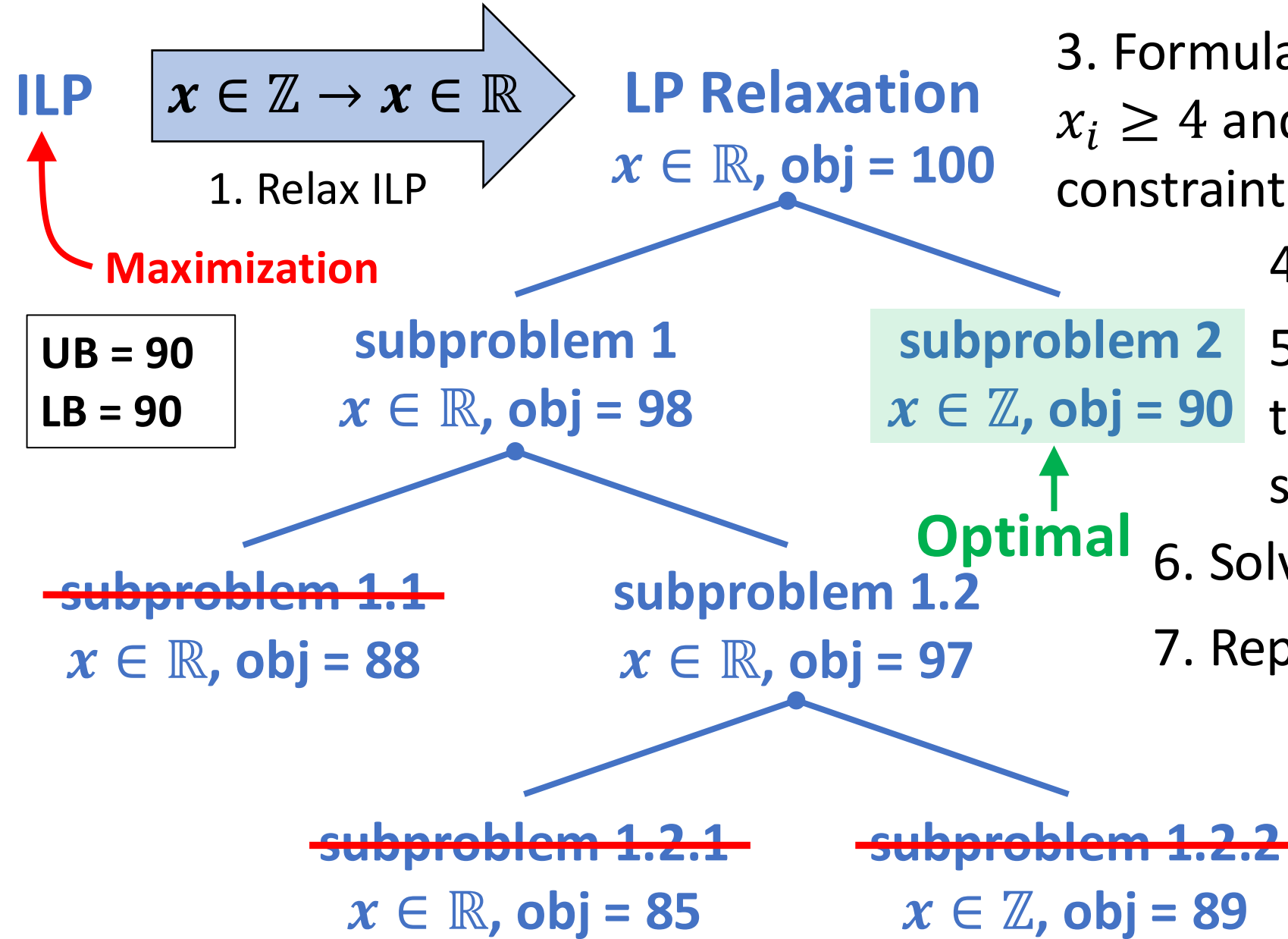
3. Formulate two new LPs. One with $x_i \geq 4$ and the other with $x_i \leq 3$ constraints.

4. Solve subproblem LPs.

5. Formulate two new LPs that split the subproblem on some non-integer variable.

6. Solve the subproblem LPs.

7. Repeat.



Total Unimodularity

Definition: A matrix is totally unimodular if the determinant of any square submatrix of it is 0, 1, or -1.

This implies that totally unimodular matrices are composed of only 0's, 1's, and -1's, since single elements are square submatrices.

It also means that if A is totally unimodular, A^T is too, since $\det(B) = \det(B^T)$.

Total Unimodularity

Definition: A matrix is totally unimodular if the determinant of any square submatrix of it is 0, 1, or -1.

Theorem (Hoffman-Kruskal, 1956):

$A \in \mathbb{R}^{m \times n}$ is
totally unimodular \iff

The feasible region:
 $\{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ has
integer vertices $\forall b \in \mathbb{Z}^m$.

Theorem (Ghouila-Houri, 1962):

$A \in \mathbb{R}^{m \times n}$ is
totally unimodular \iff

For every subset of rows R , there is a partition
 $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Vertex Cover

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

Objective: $\max c^T x$

Subject to: $Ax \leq b$

$x \geq 0$

Theorem (Ghouila-Houri, 1962):

$A \in \mathbb{R}^{m \times n}$ is
totally unimodular



For every subset of rows R , there is a partition
 $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Vertex Cover

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

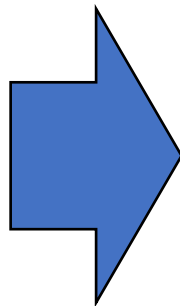
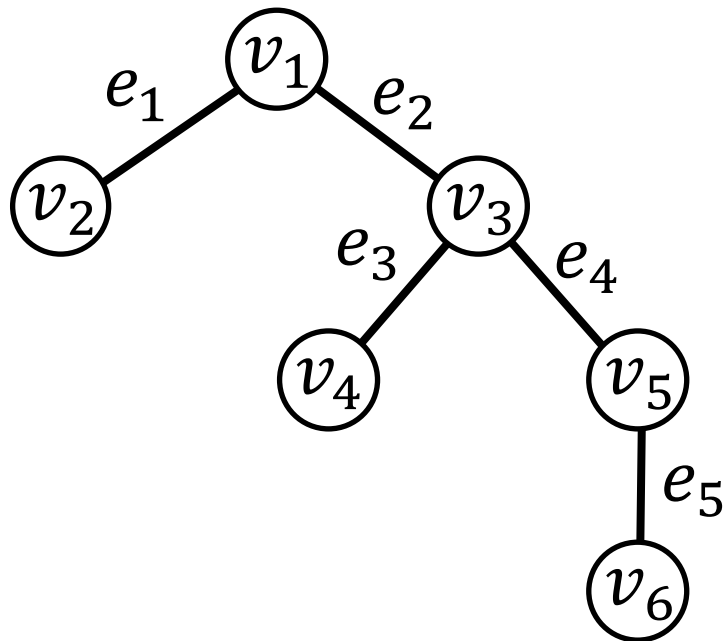
Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i,j)$

Objective: $\max c^T x$

Subject to: $A x \leq b$

$x \geq 0$



$$A = \begin{matrix} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{matrix} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{matrix}$$

Total Unimodularity

Definition: A matrix is totally unimodular if the determinant of any square submatrix of it is 0, 1, or -1.

This implies that totally unimodular matrices are composed of only 0's, 1's, and -1's, since single elements are square submatrices.

It also means that if A is totally unimodular, A^T is too, since $\det(B) = \det(B^T)$.

Vertex Cover

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

$$A = \begin{matrix} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ e_1 & \left(\begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array} \right) \end{matrix} \quad \Rightarrow \quad A^T = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 \\ x_1 & \left(\begin{array}{ccccc} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right) \end{matrix}$$

Vertex Cover

$$A^T = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 \\ x_1 & \left(\begin{array}{ccccc} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right) \\ x_2 & \\ x_3 & \\ x_4 & \\ x_5 & \\ x_6 & \end{matrix}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

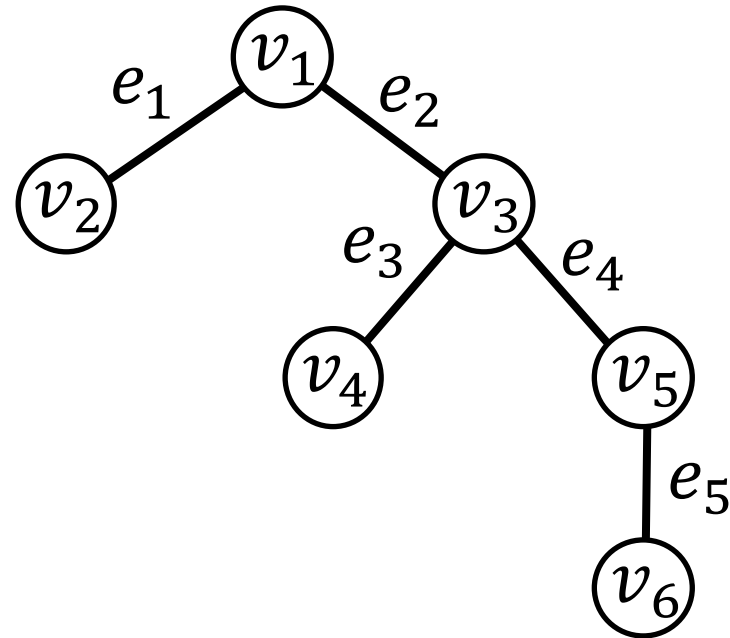
Vertex Cover

$$A^T = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 \\ x_1 & \left(\begin{array}{ccccc} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right) \end{matrix}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Label alternating generations of vertices into two sets B and G .



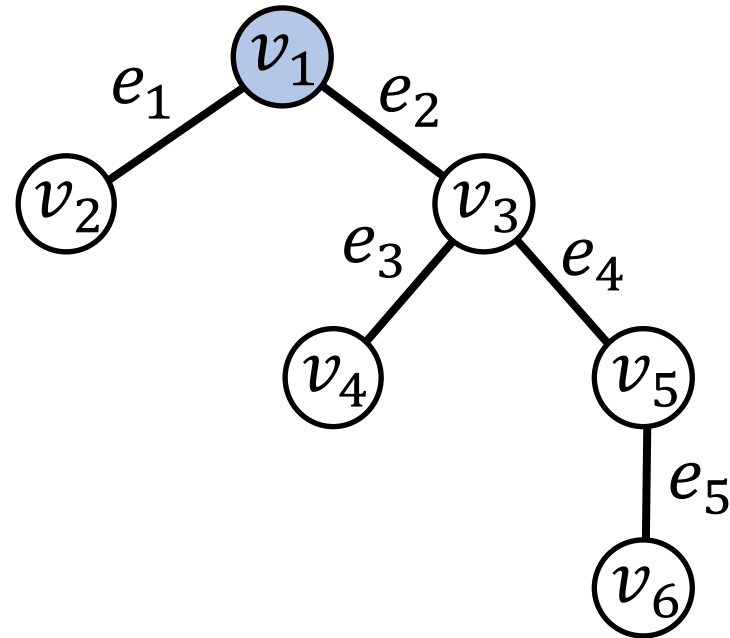
Vertex Cover

$$A^T = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{matrix} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Label alternating generations of vertices into two sets B and G .



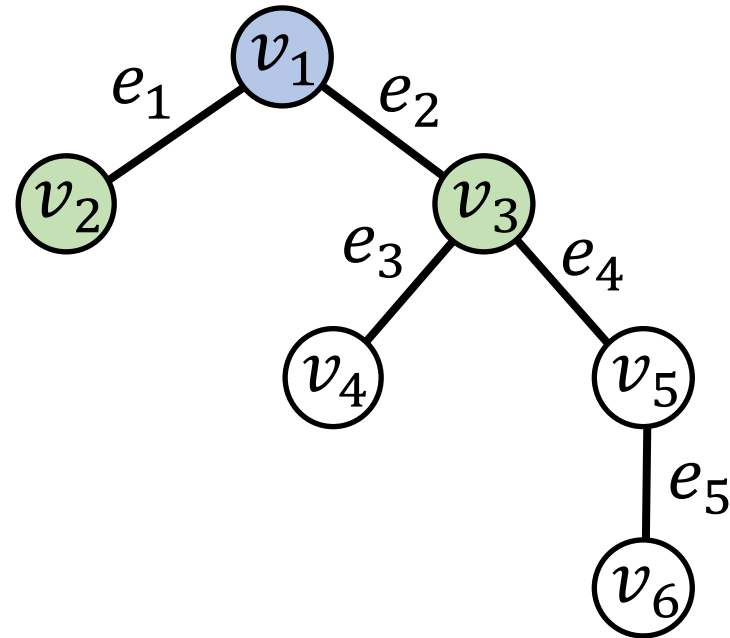
Vertex Cover

$$A^T = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{matrix} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Label alternating generations of vertices into two sets B and G .



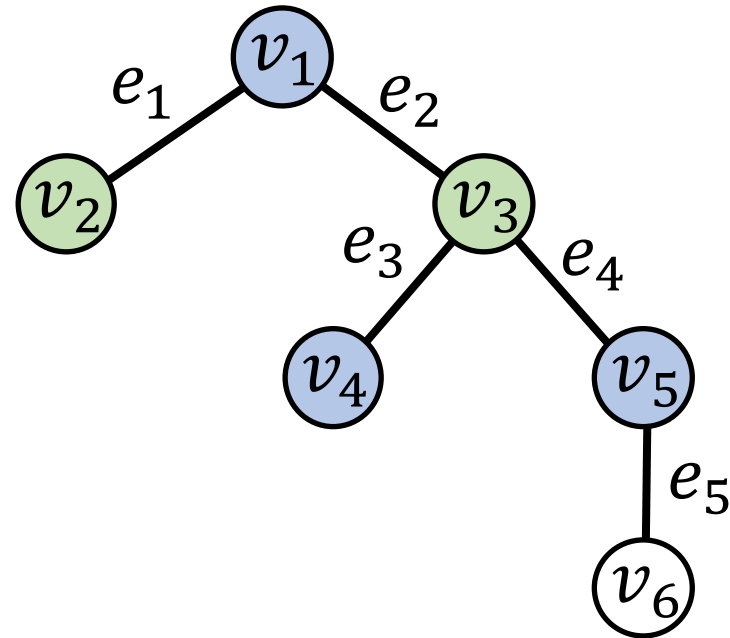
Vertex Cover

$$A^T = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{matrix} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Label alternating generations of vertices into two sets B and G .



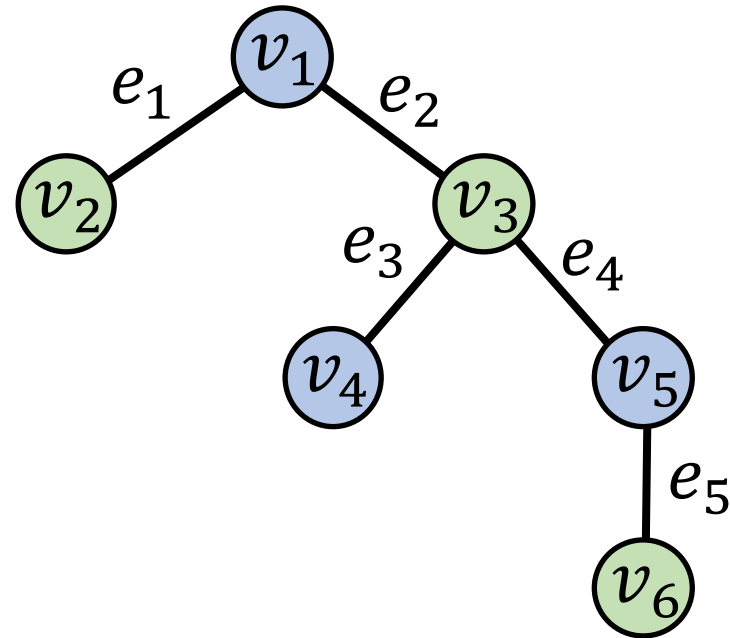
Vertex Cover

$$A^T = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Label alternating generations of vertices into two sets B and G .



Vertex Cover

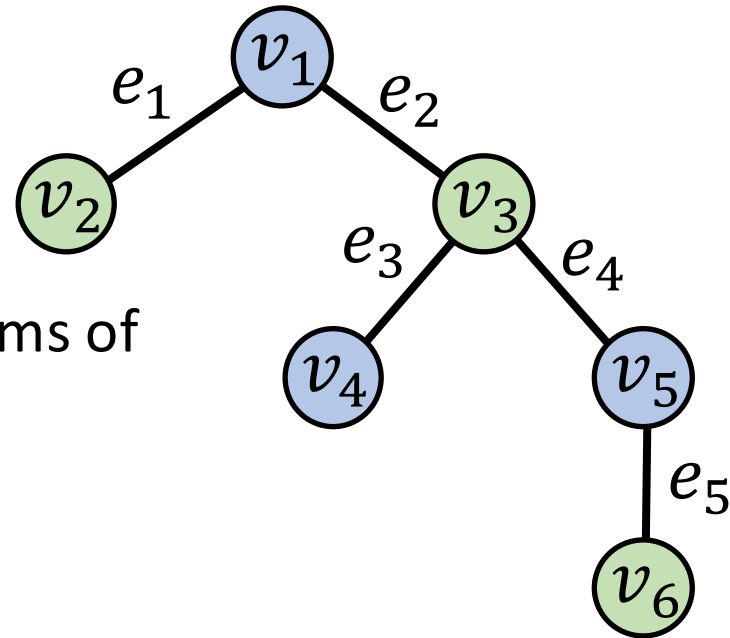
$$A^T = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}$$

What can we observe about a single column in terms of 0/1 values and B/G labels?

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Label alternating generations of vertices into two sets B and G .



Vertex Cover

$$A^T = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}$$

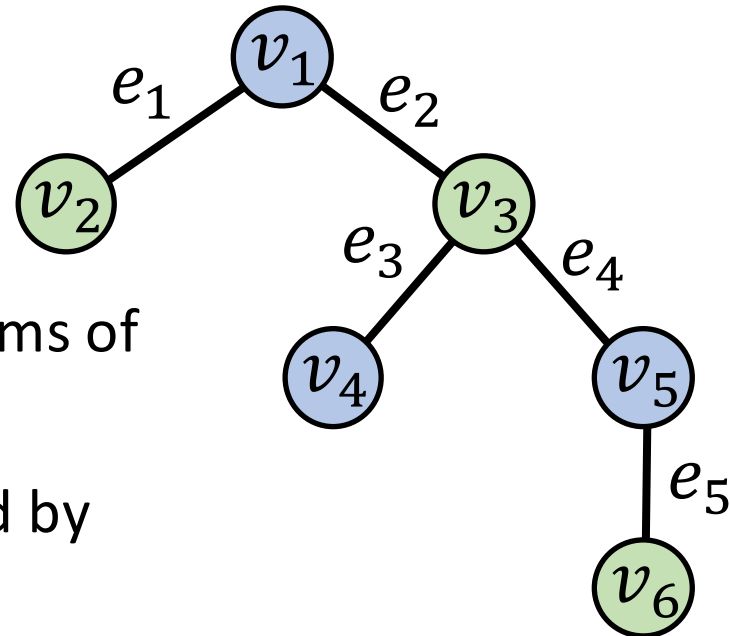
What can we observe about a single column in terms of 0/1 values and B/G labels?

Each column has exactly two 1's (edges are formed by exactly two vertices)

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Label alternating generations of vertices into two sets B and G .



Vertex Cover

$$A^T = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ \begin{array}{ccccc} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array} \end{array}$$

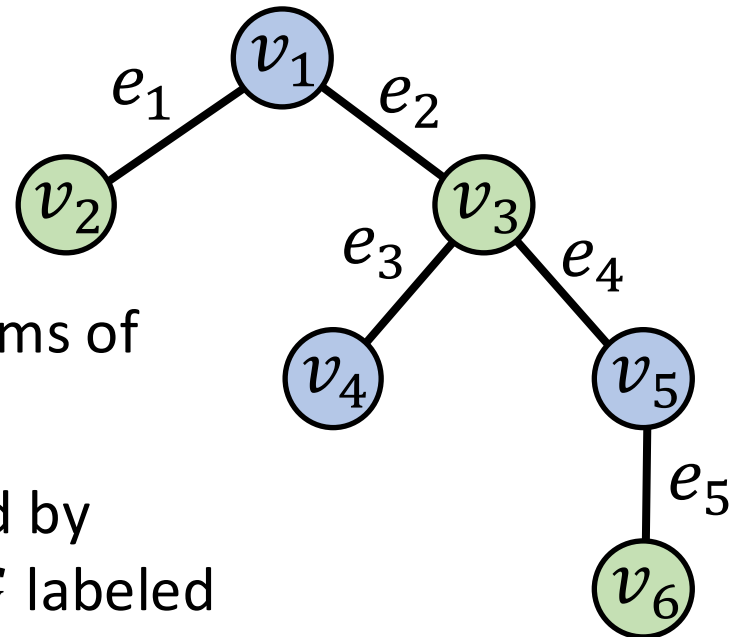
What can we observe about a single column in terms of 0/1 values and B/G labels?

Each column has exactly two 1's (edges are formed by exactly two vertices) with exactly one B and one G labeled vertex (generations don't overlap).

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Label alternating generations of vertices into two sets B and G .



Vertex Cover

$$A^T = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ \begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} & \begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{array} \end{array}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Label alternating generations of vertices into two sets B and G .

Vertex Cover

$$A^T = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Label alternating generations of vertices into two sets B and G . Consider a subset of rows of A^T .

Vertex Cover

$$A^T = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Label alternating generations of vertices into two sets B and G . Consider a subset of rows of A^T . Partition R into $R_1 = R \cap B$ and $R_2 = R \cap G$.

Vertex Cover

$$A^T = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}$$

Each column has exactly two 1's (edges are formed by exactly two vertices) with exactly one B and one G labeled vertex (generations don't overlap).

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Label alternating generations of vertices into two sets B and G . Consider a subset of rows of A^T . Partition R into $R_1 = R \cap B$ and $R_2 = R \cap G$.

Vertex Cover

$$A^T = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}$$

For any column of A^T , which corresponds to an edge e ,

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Label alternating generations of vertices into two sets B and G . Consider a subset of rows of A^T . Partition R into $R_1 = R \cap B$ and $R_2 = R \cap G$.

Vertex Cover

$$A^T = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{matrix} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Label alternating generations of vertices into two sets B and G . Consider a subset of rows of A^T . Partition R into $R_1 = R \cap B$ and $R_2 = R \cap G$.

For any column of A^T , which corresponds to an edge e ,

$$\sum_{v \in R_1} A_{ve} = ?$$

For a fixed column e , how many B labeled vertices are there?

Vertex Cover

$$A^T = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Label alternating generations of vertices into two sets B and G . Consider a subset of rows of A^T . Partition R into $R_1 = R \cap B$ and $R_2 = R \cap G$.

For any column of A^T , which corresponds to an edge e ,

$$\sum_{v \in R_1} A_{ve} \in \{0, 1\}$$

For a fixed column e , how many B labeled vertices are there?

Vertex Cover

$$A^T = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

For any column of A^T , which corresponds to an edge e ,

$$\sum_{v \in R_1} A_{ve} \in \{0,1\} \quad \text{and} \quad \sum_{v \in R_2} A_{ve} = ?$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Label alternating generations of vertices into two sets B and G . Consider a subset of rows of A^T . Partition R into $R_1 = R \cap B$ and $R_2 = R \cap G$.

For a fixed column e , how many G labeled vertices are there?

Vertex Cover

$$A^T = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}$$

For any column of A^T , which corresponds to an edge e ,

$$\sum_{v \in R_1} A_{ve} \in \{0,1\} \quad \text{and} \quad \sum_{v \in R_2} A_{ve} \in \{0,1\}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Label alternating generations of vertices into two sets B and G . Consider a subset of rows of A^T . Partition R into $R_1 = R \cap B$ and $R_2 = R \cap G$.

For a fixed column e , how many G labeled vertices are there?

Vertex Cover

$$A^T = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Label alternating generations of vertices into two sets B and G . Consider a subset of rows of A^T . Partition R into $R_1 = R \cap B$ and $R_2 = R \cap G$.

For any column of A^T , which corresponds to an edge e ,

$$\sum_{v \in R_1} A_{ve} \in \{0, 1\} \quad \text{and} \quad \sum_{v \in R_2} A_{ve} \in \{0, 1\} \quad \Rightarrow \quad \sum_{v \in R_1} A_{ve} - \sum_{v \in R_2} A_{ve} = ?$$

Vertex Cover

$$A^T = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Label alternating generations of vertices into two sets B and G . Consider a subset of rows of A^T . Partition R into $R_1 = R \cap B$ and $R_2 = R \cap G$.

For any column of A^T , which corresponds to an edge e ,

$$\sum_{v \in R_1} A_{ve} \in \{0, 1\} \quad \text{and} \quad \sum_{v \in R_2} A_{ve} \in \{0, 1\} \quad \Rightarrow \quad \sum_{v \in R_1} A_{ve} - \sum_{v \in R_2} A_{ve} \in \{-1, 0, 1\}$$

Vertex Cover

$$A^T = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{matrix} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

For any column of A^T , which corresponds to an edge e ,

$$\sum_{v \in R_1} A_{ve} \in \{0,1\} \quad \text{and} \quad \sum_{v \in R_2} A_{ve} \in \{0,1\} \quad \Rightarrow \quad \sum_{v \in R_1} A_{ve} - \sum_{v \in R_2} A_{ve} \in \{-1,0,1\}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1,0,1\}$$

Label alternating generations of vertices into two sets B and G . Consider a subset of rows of A^T . Partition R into $R_1 = R \cap B$ and $R_2 = R \cap G$.

Therefore, A^T (and thus A) is totally unimodular.

Vertex Cover

$$A^T = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{matrix} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

$x_i \in \{0, 1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

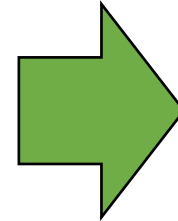
Objective: $\max c^T x$
Subject to: $Ax \leq b$
 $x \geq 0$

For any column of A^T , which corresponds to an edge e ,

$$\sum_{v \in R_1} A_{ve} \in \{0, 1\} \quad \text{and} \quad \sum_{v \in R_2} A_{ve} \in \{0, 1\} \quad \Rightarrow \quad \sum_{v \in R_1} A_{ve} - \sum_{v \in R_2} A_{ve} \in \{-1, 0, 1\}$$

LP Standard Form

$$\begin{aligned} \text{Objective: } & \max 100x_1 + 300x_2 \\ \text{Subject to: } & x_1 \leq 30 \\ & x_2 \leq 20 \\ & x_1 + x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$



$$\begin{aligned} \text{Objective: } & \max c^T x \\ \text{Subject to: } & A x \leq b \\ & x \geq 0 \end{aligned}$$

Diagram illustrating the mapping of the specific LP to the standard form:

- The objective coefficients $(100, 300)$ are mapped to c^T .
- The decision variables (x_1, x_2) are mapped to x .
- The constraint matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ is shown below the standard form.
- The constraint vector $b = \begin{pmatrix} 30 \\ 20 \\ 40 \end{pmatrix}$ is shown below the standard form.

Every LP can be turned into standard form.

1. Minimization \rightarrow Maximization: Multiply objective coefficients by -1.

$$\min \alpha x_1 + \beta x_2 \rightarrow \max -\alpha x_1 - \beta x_2$$

2. \geq Constraints $\rightarrow \leq$: Negate inequality.

$$x_1 + x_2 \geq \alpha \rightarrow -x_1 - x_2 \leq -\alpha$$

3. Equality Constraints $\rightarrow \leq$: ?

4. Unrestricted sign $x_1 \rightarrow x_1 \geq 0$:

Vertex Cover

$$-A^T = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 \\ x_1 & -1 & -1 & 0 & 0 & 0 \\ x_2 & -1 & 0 & 0 & 0 & 0 \\ x_3 & 0 & -1 & -1 & -1 & 0 \\ x_4 & 0 & 0 & -1 & 0 & 0 \\ x_5 & 0 & 0 & 0 & -1 & -1 \\ x_6 & 0 & 0 & 0 & 0 & -1 \end{matrix}$$

For any column of A^T , which corresponds to an edge e ,

$$\sum_{v \in R_1} A_{ve} \in \{0,1\} \quad \text{and} \quad \sum_{v \in R_2} A_{ve} \in \{0,1\} \quad \Rightarrow \quad \sum_{v \in R_1} A_{ve} - \sum_{v \in R_2} A_{ve} \in \{-1,0,1\}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1,0,1\}$$

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

Objective: $\max c^T x$
Subject to: $Ax \leq b$
 $x \geq 0$

Vertex Cover

$$-A^T = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 \\ x_1 & -1 & -1 & 0 & 0 & 0 \\ x_2 & -1 & 0 & 0 & 0 & 0 \\ x_3 & 0 & -1 & -1 & -1 & 0 \\ x_4 & 0 & 0 & -1 & 0 & 0 \\ x_5 & 0 & 0 & 0 & -1 & -1 \\ x_6 & 0 & 0 & 0 & 0 & -1 \end{matrix}$$

For any column of A^T , which corresponds to an edge e ,

$$\sum_{v \in R_1} A_{ve} \in \{0,1\} \text{ and } \sum_{v \in R_2} A_{ve} \in \{0,1\} \Rightarrow \sum_{v \in R_1} A_{ve} - \sum_{v \in R_2} A_{ve} \in \{-1,0,1\}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1,0,1\}$$

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

Objective: $\max c^T x$
Subject to: $Ax \leq b$
 $x \geq 0$

Vertex Cover

$$-A^T = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{matrix} & \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \end{matrix}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

$x_i \in \{0, 1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

Objective: $\max c^T x$
Subject to: $Ax \leq b$
 $x \geq 0$

For any column of A^T , which corresponds to an edge e ,

$$\sum_{v \in R_1} A_{ve} \in \{-1, 0\} \text{ and } \sum_{v \in R_2} A_{ve} \in \{0, 1\} \Rightarrow \sum_{v \in R_1} A_{ve} - \sum_{v \in R_2} A_{ve} \in \{-1, 0, 1\}$$

Vertex Cover

$$-A^T = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 \\ x_1 & -1 & -1 & 0 & 0 & 0 \\ x_2 & -1 & 0 & 0 & 0 & 0 \\ x_3 & 0 & -1 & -1 & -1 & 0 \\ x_4 & 0 & 0 & -1 & 0 & 0 \\ x_5 & 0 & 0 & 0 & -1 & -1 \\ x_6 & 0 & 0 & 0 & 0 & -1 \end{matrix}$$

For any column of A^T , which corresponds to an edge e ,

$$\sum_{v \in R_1} A_{ve} \in \{-1, 0\} \text{ and } \sum_{v \in R_2} A_{ve} \in \{-1, 0\} \Rightarrow \sum_{v \in R_1} A_{ve} - \sum_{v \in R_2} A_{ve} \in \{-1, 0, 1\}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

$x_i \in \{0, 1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

Objective: $\max c^T x$
 Subject to: $Ax \leq b$
 $x \geq 0$

Vertex Cover

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

Objective: $\max c^T x$

Subject to: $A x \leq b$

$x \geq 0$

Theorem (Hoffman-Kruskal, 1956):

$A \in \mathbb{R}^{m \times n}$ is
totally unimodular \iff

The feasible region:
 $\{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ has
integer vertices $\forall b \in \mathbb{Z}^m$.

\therefore The Vertex Cover ILP will be solved in polynomial time when the graph is a tree.

Vertex Cover

$$A^T = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{matrix} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Label alternating generations of vertices into two sets B and G . Consider a subset of rows of A^T . Partition R into $R_1 = R \cap B$ and $R_2 = R \cap G$.

For any column of A^T , which corresponds to an edge e ,

$$\sum_{v \in R_1} A_{ve} \in \{0, 1\} \quad \text{and} \quad \sum_{v \in R_2} A_{ve} \in \{0, 1\} \quad \Rightarrow \quad \sum_{v \in R_1} A_{ve} - \sum_{v \in R_2} A_{ve} \in \{-1, 0, 1\}$$

Does this work for any other special graphs beyond trees?

Vertex Cover

$$A^T = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{matrix} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Bipartite!

Label alternating generations of vertices into two sets B and G . Consider a subset of rows of A^T . Partition R into $R_1 = R \cap B$ and $R_2 = R \cap G$.

For any column of A^T , which corresponds to an edge e ,

$$\sum_{v \in R_1} A_{ve} \in \{0, 1\} \quad \text{and} \quad \sum_{v \in R_2} A_{ve} \in \{0, 1\} \quad \Rightarrow \quad \sum_{v \in R_1} A_{ve} - \sum_{v \in R_2} A_{ve} \in \{-1, 0, 1\}$$

Does this work for any other special graphs beyond trees?

Vertex Cover

$x_i \in \{0,1\}$ = Indicates if vertex i is selected.

Objective: $\min \sum_i x_i$

Subject to: $x_i + x_j \geq 1$, for each edge $e = (i, j)$

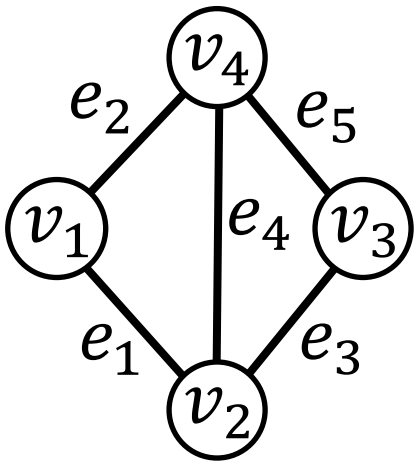
But they all use
the same ILP!

General Graphs: NP-Hard

Trees: $O(n)$

Bipartite: Polynomial

Vertex Cover (General Graph)



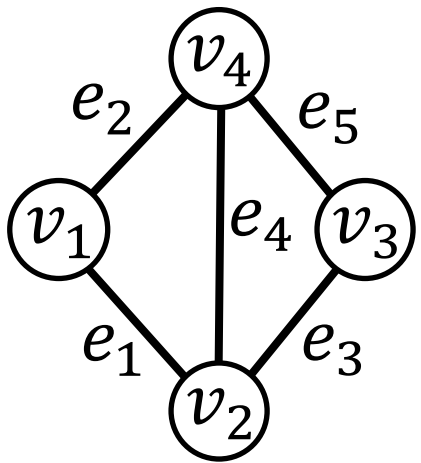
Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

Vertex Cover (General Graph)

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$

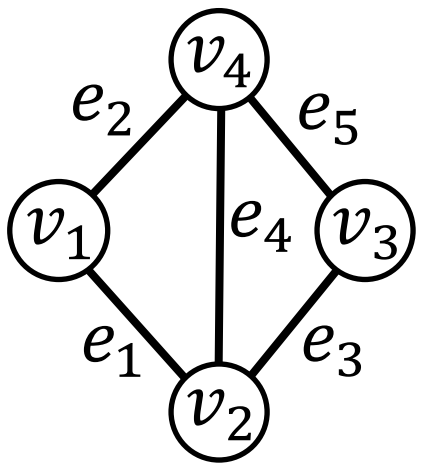


$$A = \begin{matrix} & x_1 & x_2 & x_3 & x_4 \\ e_1 & \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ e_2 & \\ e_3 & \\ e_4 & \\ e_5 & \end{matrix}$$

Vertex Cover (General Graph)

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$



$A =$

$$\begin{matrix} & x_1 & x_2 & x_3 & x_4 \\ e_1 & \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{matrix}$$



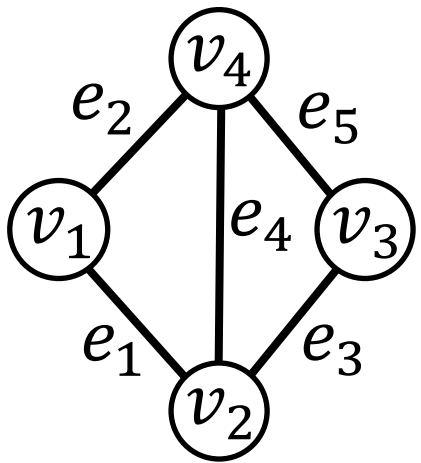
$A^T =$

$$\begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 \\ x_1 & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \\ x_2 \\ x_3 \\ x_4 \end{matrix}$$

Vertex Cover (General Graph)

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$



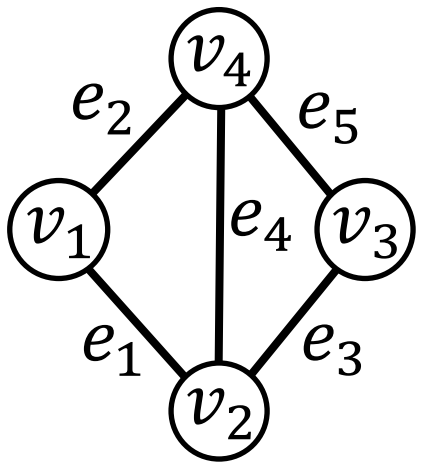
$$A^T = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 \\ x_1 & 1 & 1 & 0 & 0 & 0 \\ x_2 & 1 & 0 & 1 & 1 & 0 \\ x_3 & 0 & 0 & 1 & 0 & 1 \\ x_4 & 0 & 1 & 0 & 1 & 1 \end{matrix}$$

Where is the contradiction with Ghouila-Houri?

Vertex Cover (General Graph)

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$



$A^T =$

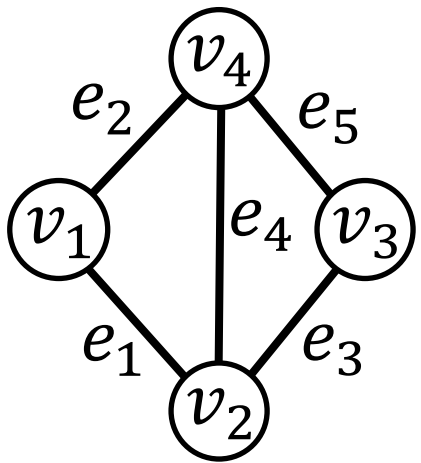
	e_1	e_2	e_3	e_4	e_5
x_1	1	1	0	0	0
x_2	1	0	1	1	0
x_3	0	0	1	0	1
x_4	0	1	0	1	1

Where is the contradiction with Ghouila-Houri?

Vertex Cover (General Graph)

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$



$A^T =$

	e_1	e_2	e_3	e_4	e_5
x_1	1	1	0	0	0
x_2	1	0	1	1	0
x_3	0	0	1	0	1
x_4	0	1	0	1	1

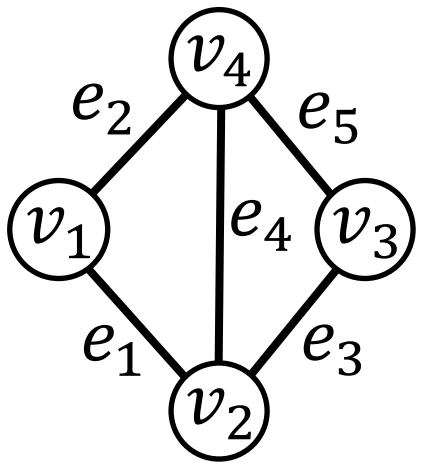
Where is the contradiction with Ghouila-Houri?

x_1 and x_2 have to be on different sides of the partition.

Vertex Cover (General Graph)

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$



$A^T =$

	e_1	e_2	e_3	e_4	e_5
x_1	1	1	0	0	0
x_2	1	0	1	1	0
x_3	0	0	1	0	1
x_4	0	1	0	1	1

Where is the contradiction with Ghouila-Houri?

x_1 and x_2 have to be on different sides of the partition.

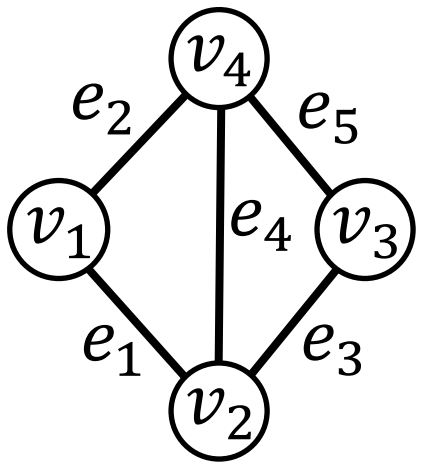
$$R_1 = \{x_1\}$$

$$R_2 = \{x_2\}$$

Vertex Cover (General Graph)

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$



$A^T =$

	e_1	e_2	e_3	e_4	e_5
x_1	1	1	0	0	0
x_2	1	0	1	1	0
x_3	0	0	1	0	1
x_4	0	1	0	1	1

Where is the contradiction with Ghouila-Houri?

x_1 and x_2 have to be on different sides of the partition.

$$R_1 = \{x_1, x_4\}$$

$$R_2 = \{x_2\}$$

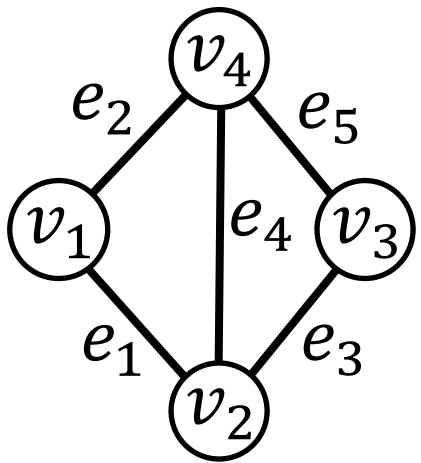
$$R_1 = \{x_1\}$$

$$R_2 = \{x_2, x_4\}$$

Vertex Cover (General Graph)

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$



$A^T =$

	e_1	e_2	e_3	e_4	e_5
x_1	1	1	0	0	0
x_2	1	0	1	1	0
x_3	0	0	1	0	1
x_4	0	1	0	1	1

Where is the contradiction with Ghouila-Houri?

x_1 and x_2 have to be on different sides of the partition.

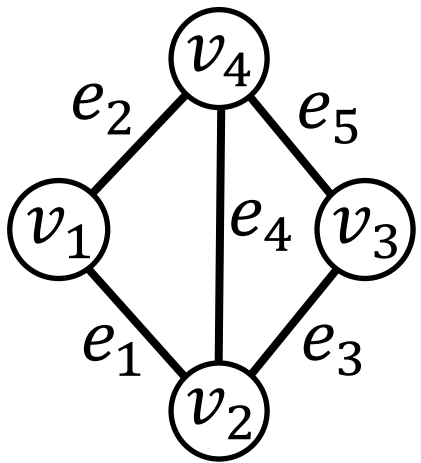
$$\left. \begin{array}{l} R_1 = \{x_1, x_4\} \\ R_2 = \{x_2\} \end{array} \right\} \text{Column } e_2 \text{ sums to 2.}$$

$$\begin{array}{l} R_1 = \{x_1\} \\ R_2 = \{x_2, x_4\} \end{array}$$

Vertex Cover (General Graph)

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$



$A^T =$

	e_1	e_2	e_3	e_4	e_5
x_1	1	1	0	0	0
x_2	1	0	1	1	0
x_3	0	0	1	0	1
x_4	0	1	0	1	1

Where is the contradiction with Ghouila-Houri?

x_1 and x_2 have to be on different sides of the partition.

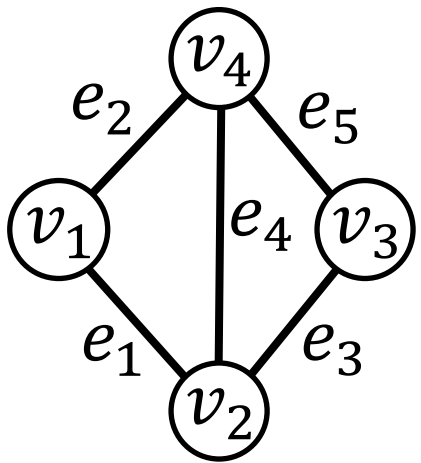
$$\left. \begin{array}{l} R_1 = \{x_1, x_4\} \\ R_2 = \{x_2\} \end{array} \right\} \text{Column } e_2 \text{ sums to 2.}$$

$$\left. \begin{array}{l} R_1 = \{x_1\} \\ R_2 = \{x_2, x_4\} \end{array} \right\} \text{Column } e_4 \text{ sums to -2.}$$

Vertex Cover (General Graph)

Theorem (Ghouila-Houri, 1962): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow For every subset of rows R , there is a partition $R = R_1 \cup R_2$ such that for every column j ,

$$\sum_{i \in R_1} A_{ij} - \sum_{i \in R_2} A_{ij} \in \{-1, 0, 1\}$$



$A^T =$

	e_1	e_2	e_3	e_4	e_5
x_1	1	1	0	0	0
x_2	1	0	1	1	0
x_3	0	0	1	0	1
x_4	0	1	0	1	1

Where is the contradiction with Ghouila-Houri?

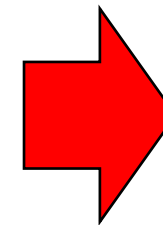
x_1 and x_2 have to be on different sides of the partition.

$R_1 = \{x_1, x_4\}$
 $R_2 = \{x_2\}$

Column e_2 sums to 2.

$R_1 = \{x_1\}$
 $R_2 = \{x_2, x_4\}$

Column e_4 sums to -2.



No such partition exists, so A^T and A are not totally unimodular.

Final Notes

Theorem (Hoffman-Kruskal, 1956): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow The feasible region: $\{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ has integer vertices $\forall b \in \mathbb{Z}^m$.

Total unimodularity is not the only route to an integer-vertexed feasible region.

Final Notes

Theorem (Hoffman-Kruskal, 1956): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow The feasible region: $\{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ has integer vertices $\forall b \in \mathbb{Z}^m$.

$$x_1, x_2 \in \mathbb{Z}$$

$$\text{Objective: } \min x_1 + x_2$$

$$\text{Subject to: } x_1 + x_2 \leq 1$$

$$x_1 - x_2 \leq 1$$

$$x_1, x_2 \geq 0$$

Total unimodularity is not the only route to an integer-vertexed feasible region.

Final Notes

Theorem (Hoffman-Kruskal, 1956): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow The feasible region: $\{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ has integer vertices $\forall b \in \mathbb{Z}^m$.

$$x_1, x_2 \in \mathbb{Z}$$

$$\text{Objective: } \min x_1 + x_2$$

$$\text{Subject to: } x_1 + x_2 \leq 1$$

$$x_1 - x_2 \leq 1$$

$$x_1, x_2 \geq 0$$



$$A = \begin{matrix} & x_1 & x_2 \\ c_1 & \begin{pmatrix} 1 & 1 \end{pmatrix} \\ c_2 & \begin{pmatrix} 1 & -1 \end{pmatrix} \\ c_3 & \begin{pmatrix} -1 & 0 \end{pmatrix} \\ c_4 & \begin{pmatrix} 0 & -1 \end{pmatrix} \end{matrix}$$

Total unimodularity is not the only route to an integer-vertexed feasible region.

Final Notes

Theorem (Hoffman-Kruskal, 1956): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow The feasible region: $\{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ has integer vertices $\forall b \in \mathbb{Z}^m$.

$$x_1, x_2 \in \mathbb{Z}$$

$$\text{Objective: } \min x_1 + x_2$$

$$\text{Subject to: } x_1 + x_2 \leq 1$$

$$x_1 - x_2 \leq 1$$

$$x_1, x_2 \geq 0$$



$$A = \begin{matrix} & x_1 & x_2 \\ c_1 & \boxed{1} & \boxed{1} \\ c_2 & \boxed{1} & \boxed{-1} \\ c_3 & -1 & 0 \\ c_4 & 0 & -1 \end{matrix}$$

determinant = -2

Total unimodularity is not the only route to an integer-vertexed feasible region.

Final Notes

Theorem (Hoffman-Kruskal, 1956): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow The feasible region: $\{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ has integer vertices $\forall b \in \mathbb{Z}^m$.

$$x_1, x_2 \in \mathbb{Z}$$

$$\text{Objective: } \min x_1 + x_2$$

$$\text{Subject to: } x_1 + x_2 \leq 1$$

$$x_1 - x_2 \leq 1$$

$$x_1, x_2 \geq 0$$



$$A = \begin{matrix} & x_1 & x_2 \\ c_1 & \boxed{1} & \boxed{1} \\ c_2 & \boxed{1} & \boxed{-1} \\ c_3 & -1 & 0 \\ c_4 & 0 & -1 \end{matrix}$$

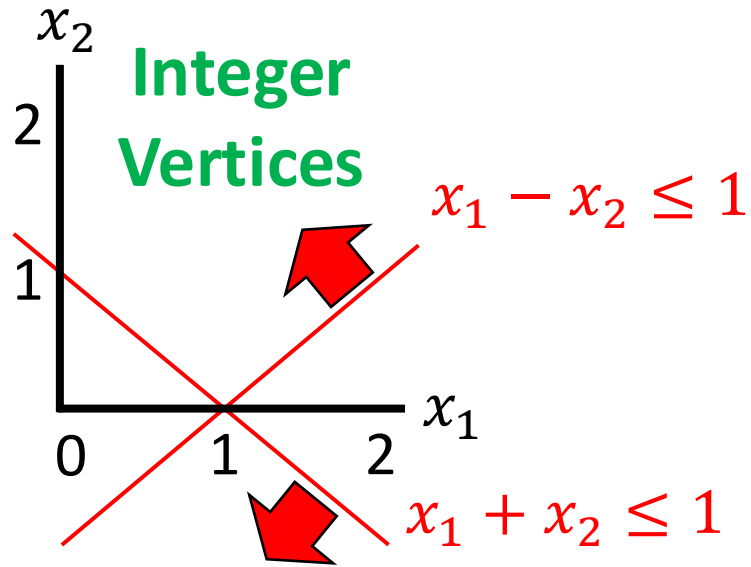
**Non-totally
Unimodular**

determinant = -2

Total unimodularity is not the only route to an integer-vertexed feasible region.

Final Notes

Theorem (Hoffman-Kruskal, 1956): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow The feasible region: $\{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ has integer vertices $\forall b \in \mathbb{Z}^m$.



$$\begin{aligned}
 &x_1, x_2 \in \mathbb{Z} \\
 &\text{Objective: } \min x_1 + x_2 \\
 &\text{Subject to: } x_1 + x_2 \leq 1 \\
 &\quad \quad \quad x_1 - x_2 \leq 1 \\
 &\quad \quad \quad x_1, x_2 \geq 0
 \end{aligned}$$

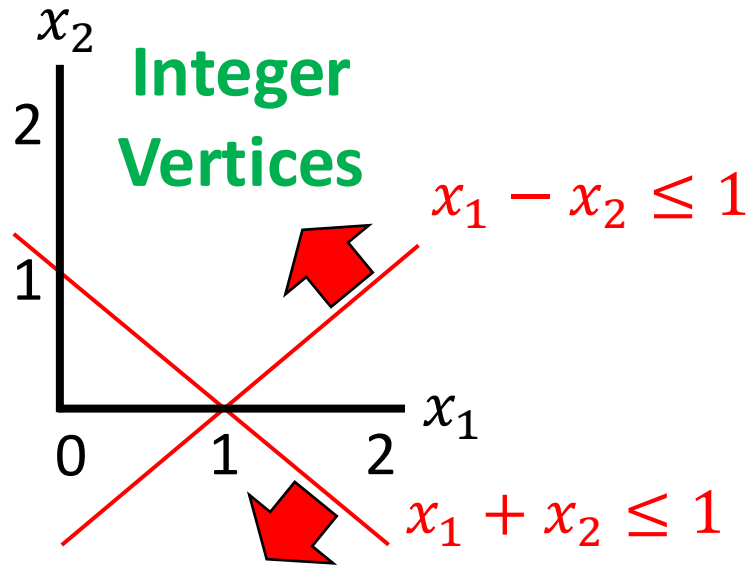
$$A = \begin{matrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{matrix} \begin{pmatrix} x_1 & x_2 \\ 1 & 1 \\ 1 & -1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Non-totally Unimodular

Total unimodularity is not the only route to an integer-vertexed feasible region.

Final Notes

Theorem (Hoffman-Kruskal, 1956): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow The feasible region: $\{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ has integer vertices $\forall b \in \mathbb{Z}^m$.



$$x_1, x_2 \in \mathbb{Z}$$

Objective: $\min x_1 + x_2$

Subject to: $x_1 + x_2 \leq 1$
 $x_1 - x_2 \leq 1$
 $x_1, x_2 \geq 0$

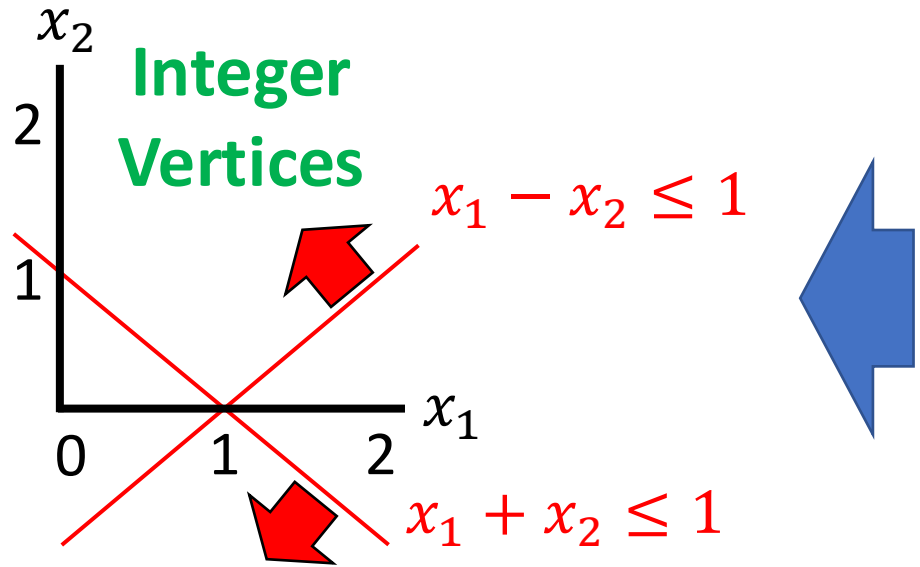
Non-totally Unimodular

$$A = \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \begin{matrix} x_1 & x_2 \\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \end{matrix}$$

Total unimodularity is not the only route to an integer-vertexed feasible region.

Final Notes

Theorem (Hoffman-Kruskal, 1956): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow The feasible region: $\{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ has integer vertices $\forall b \in \mathbb{Z}^m$.



$$\begin{aligned}
 &x_1, x_2 \in \mathbb{Z} \\
 &\text{Objective: } \min x_1 + x_2 \\
 &\text{Subject to: } x_1 + x_2 \leq 1 \\
 &\quad \quad \quad x_1 - x_2 \leq 1 \\
 &\quad \quad \quad x_1, x_2 \geq 0
 \end{aligned}$$

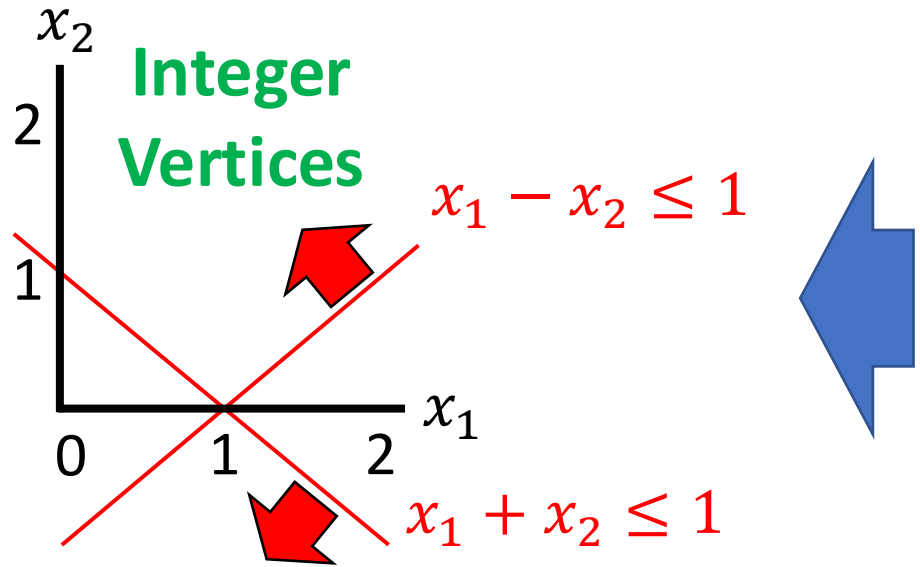
$$\begin{aligned}
 &x_1, x_2 \in \mathbb{Z} \\
 &\text{Objective: } \min x_1 + x_2 \\
 &\text{Subject to: } x_1 + x_2 \leq 1 \\
 &\quad \quad \quad x_1 - x_2 \leq 0 \\
 &\quad \quad \quad x_1, x_2 \geq 0
 \end{aligned}$$

$$A = \begin{matrix} & x_1 & x_2 \\ c_1 & \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \\ c_2 & \\ c_3 & \\ c_4 & \end{matrix}$$

Non-totally Unimodular

Final Notes

Theorem (Hoffman-Kruskal, 1956): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow The feasible region: $\{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ has integer vertices $\forall b \in \mathbb{Z}^m$.



$$\begin{aligned}
 &x_1, x_2 \in \mathbb{Z} \\
 &\text{Objective: } \min x_1 + x_2 \\
 &\text{Subject to: } x_1 + x_2 \leq 1 \\
 &\quad \quad \quad x_1 - x_2 \leq 1 \\
 &\quad \quad \quad x_1, x_2 \geq 0
 \end{aligned}$$

$$A = \begin{matrix} & x_1 & x_2 \\ c_1 & \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \\ c_2 \\ c_3 \\ c_4 \end{matrix}$$

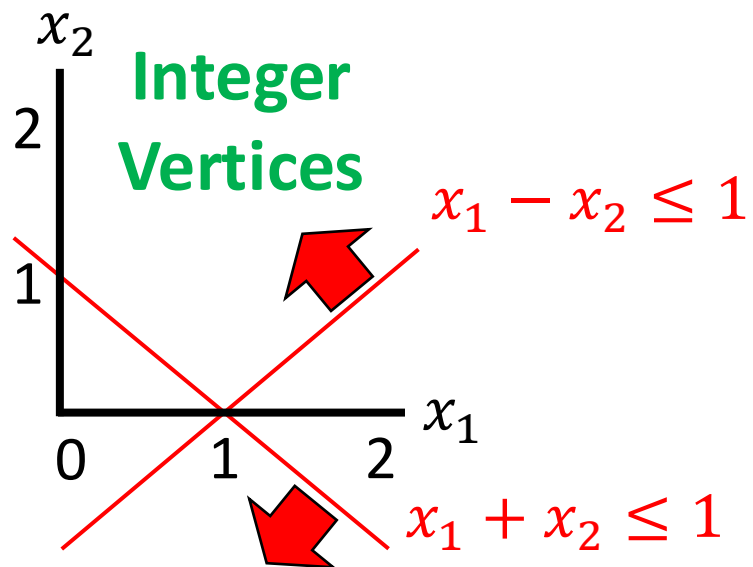
Non-totally Unimodular

$$\begin{aligned}
 &x_1, x_2 \in \mathbb{Z} \\
 &\text{Objective: } \min x_1 + x_2 \\
 &\text{Subject to: } x_1 + x_2 \leq 1 \\
 &\quad \quad \quad x_1 - x_2 \leq 0 \\
 &\quad \quad \quad x_1, x_2 \geq 0
 \end{aligned}$$

$$A = \begin{matrix} & x_1 & x_2 \\ c_1 & \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \\ c_2 \\ c_3 \\ c_4 \end{matrix}$$

Final Notes

Theorem (Hoffman-Kruskal, 1956): $A \in \mathbb{R}^{m \times n}$ is totally unimodular \Leftrightarrow The feasible region: $\{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ has integer vertices $\forall b \in \mathbb{Z}^m$.

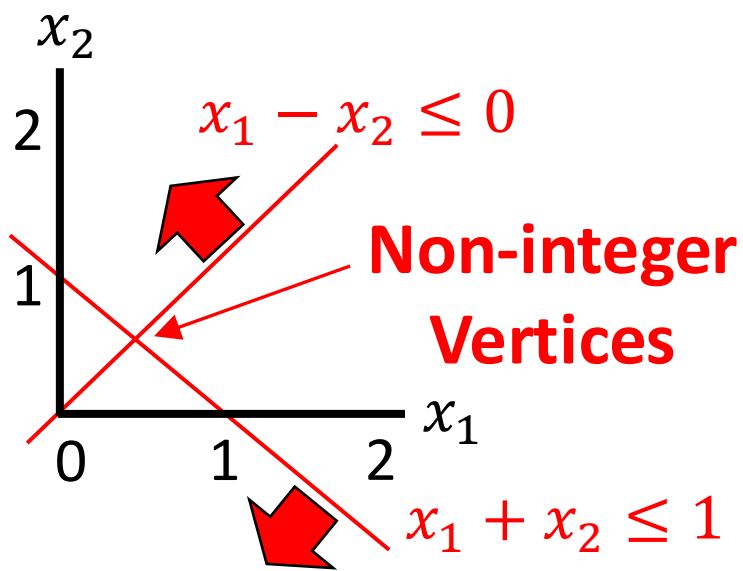


$x_1, x_2 \in \mathbb{Z}$
 Objective: $\min x_1 + x_2$
 Subject to: $x_1 + x_2 \leq 1$
 $x_1 - x_2 \leq 1$
 $x_1, x_2 \geq 0$



$$A = \begin{matrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{matrix} \begin{matrix} x_1 & x_2 \\ \left(\begin{array}{cc} 1 & 1 \\ 1 & -1 \\ -1 & 0 \\ 0 & -1 \end{array} \right) \end{matrix}$$

Non-totally Unimodular

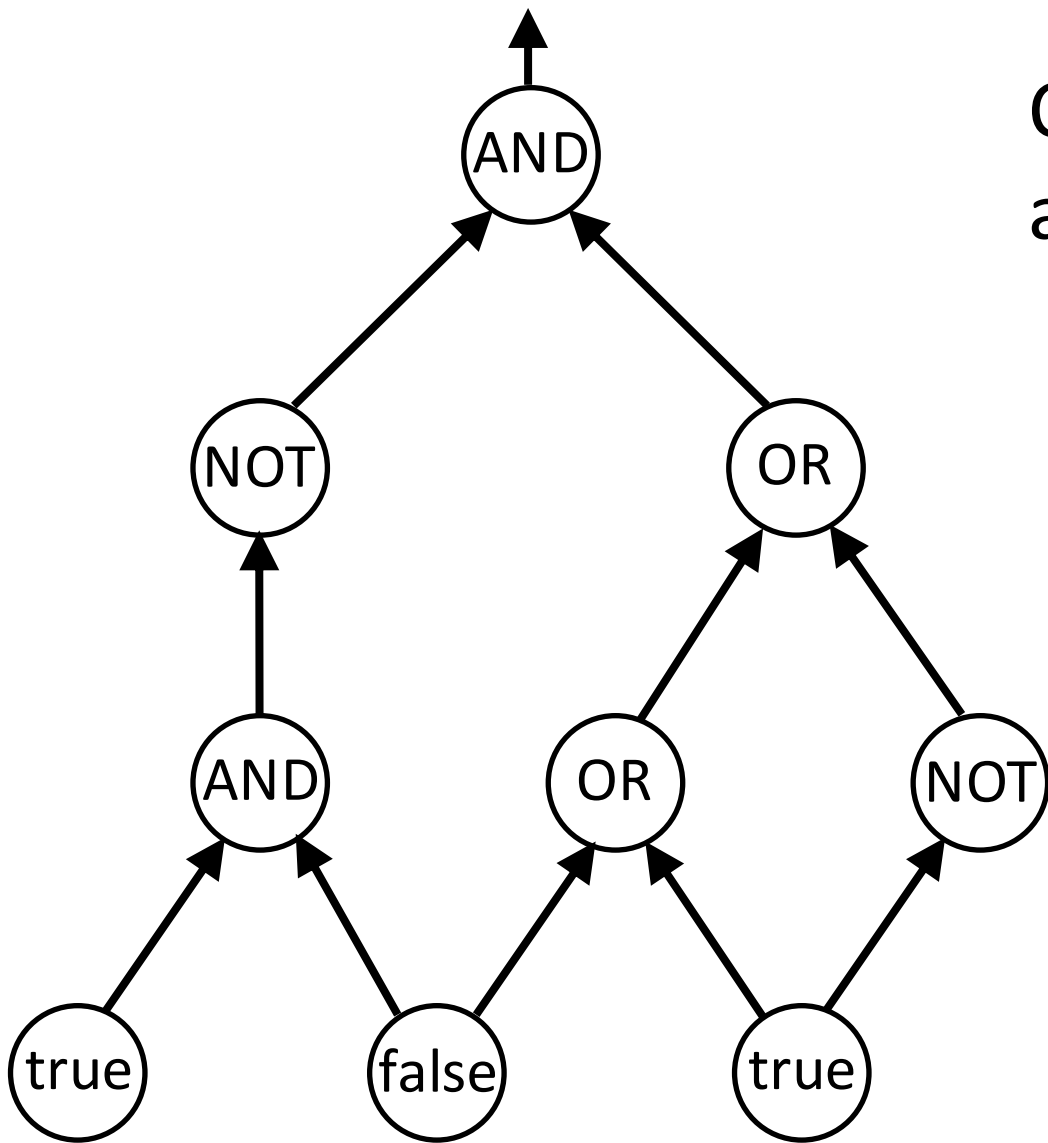


$x_1, x_2 \in \mathbb{Z}$
 Objective: $\min x_1 + x_2$
 Subject to: $x_1 + x_2 \leq 1$
 $x_1 - x_2 \leq 0$
 $x_1, x_2 \geq 0$



$$A = \begin{matrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{matrix} \begin{matrix} x_1 & x_2 \\ \left(\begin{array}{cc} 1 & 1 \\ 1 & -1 \\ -1 & 0 \\ 0 & -1 \end{array} \right) \end{matrix}$$

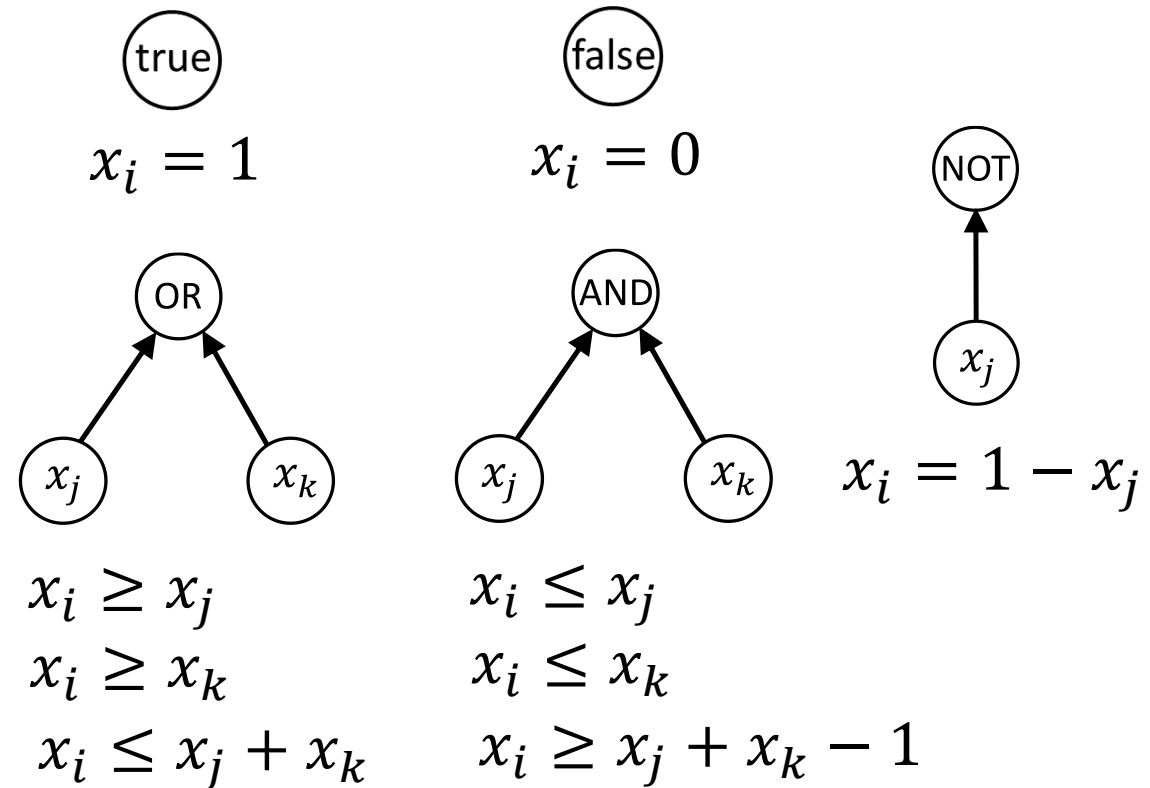
Circuit Value Problem: When the gates are evaluated, is the output true?

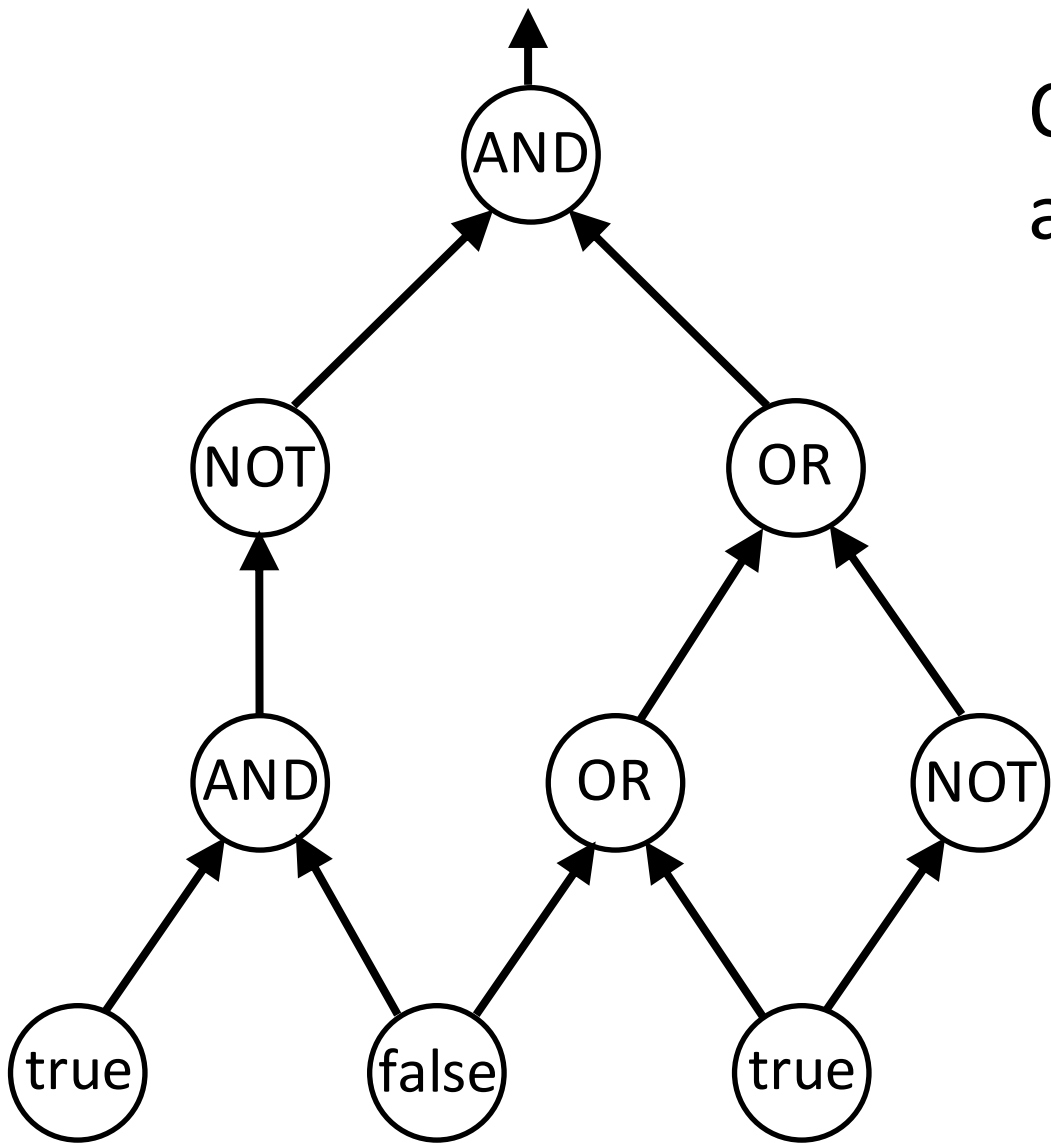


x_i = gate's evaluated value

Objective: $\min 0$

Subject to: $x_i \in \{0,1\}$





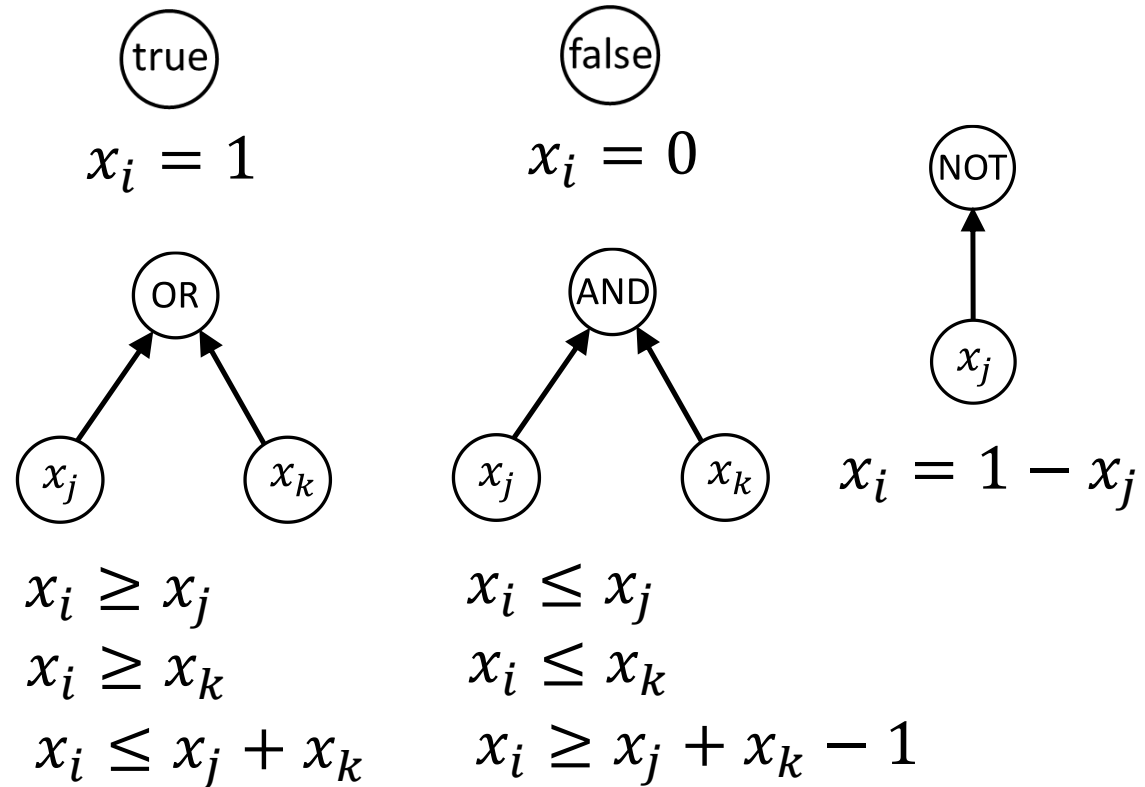
Circuit Value Problem: When the gates are evaluated, is the output true?

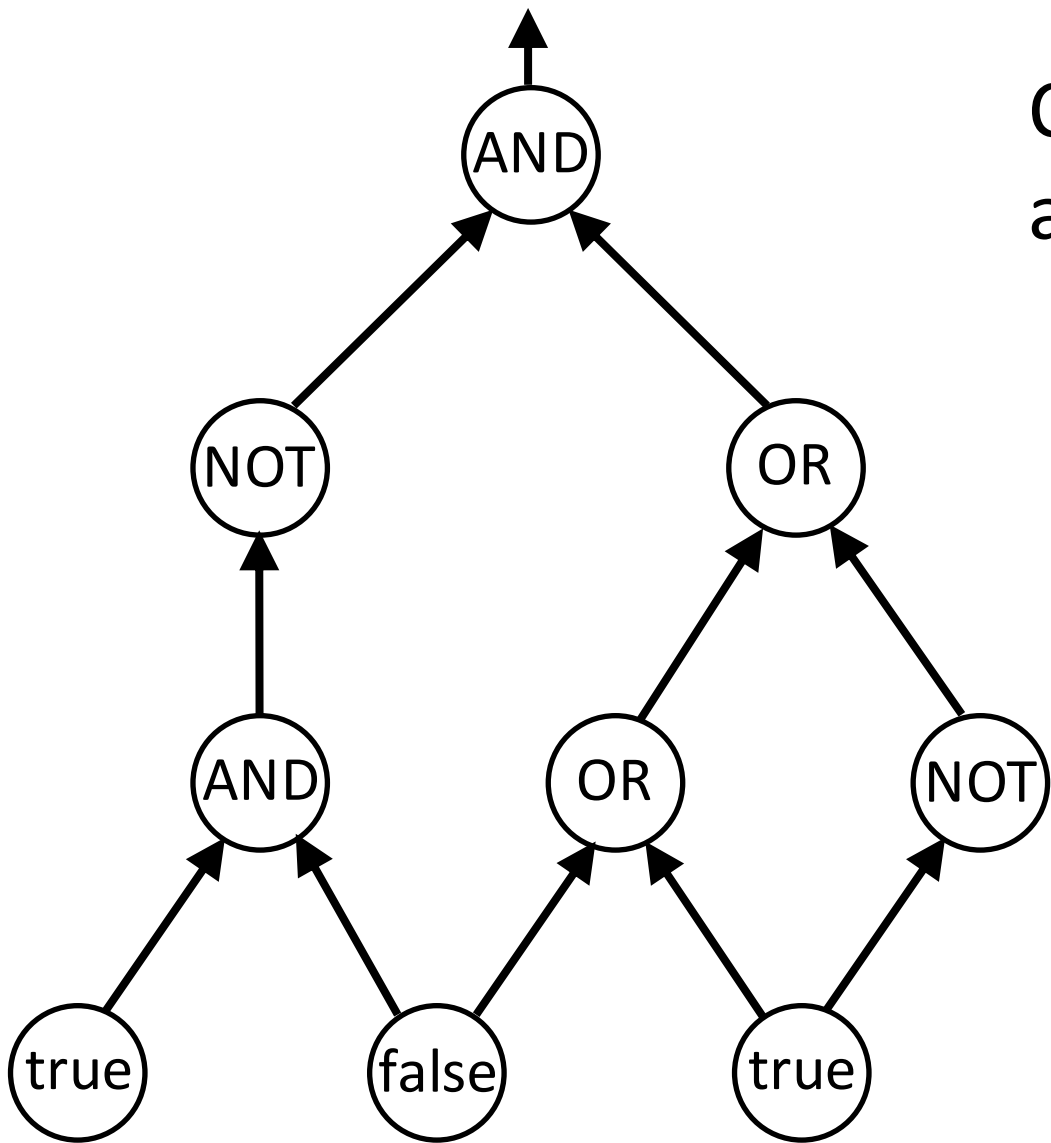
x_i = gate's evaluated value

Objective: $\min 0$

Subject to: $x_i \in [0,1]$

Will answer be an integer?





Circuit Value Problem: When the gates are evaluated, is the output true?

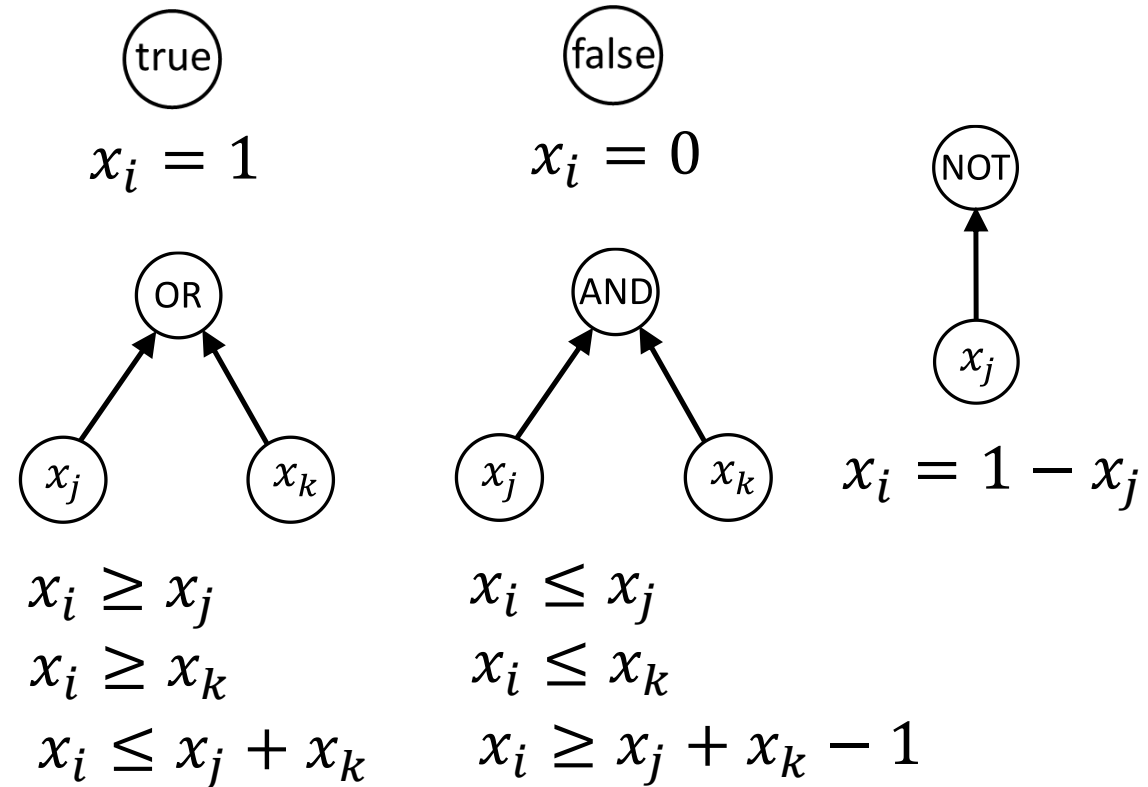
$x_i =$ gate's evaluated value

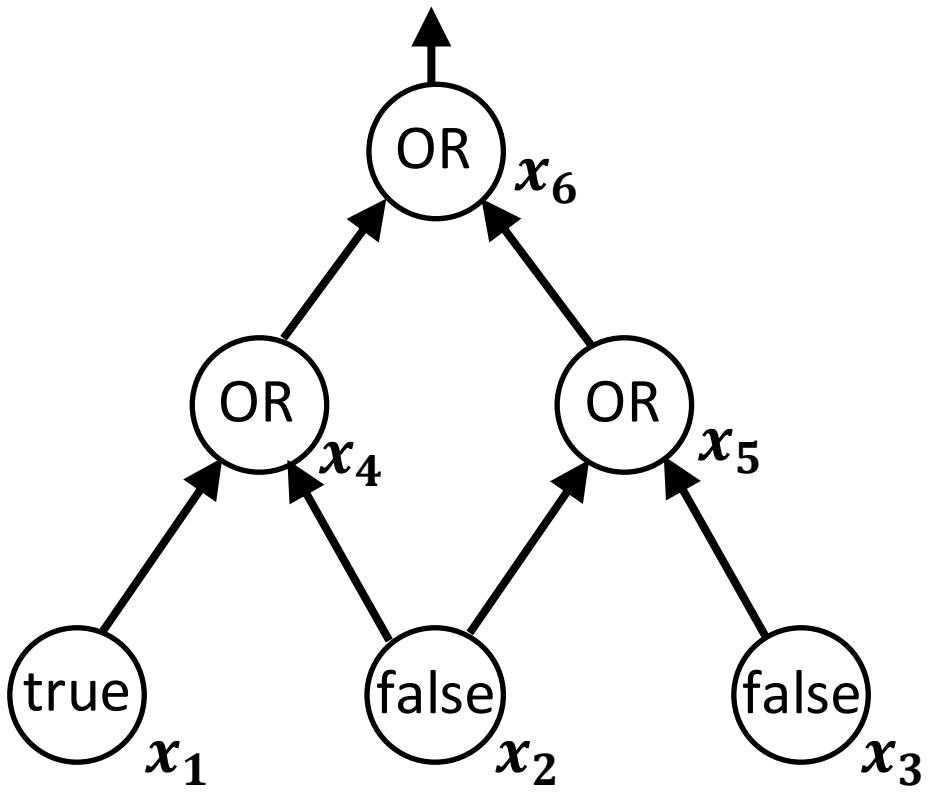
Objective: $\min 0$

Subject to: $x_i \in [0,1]$

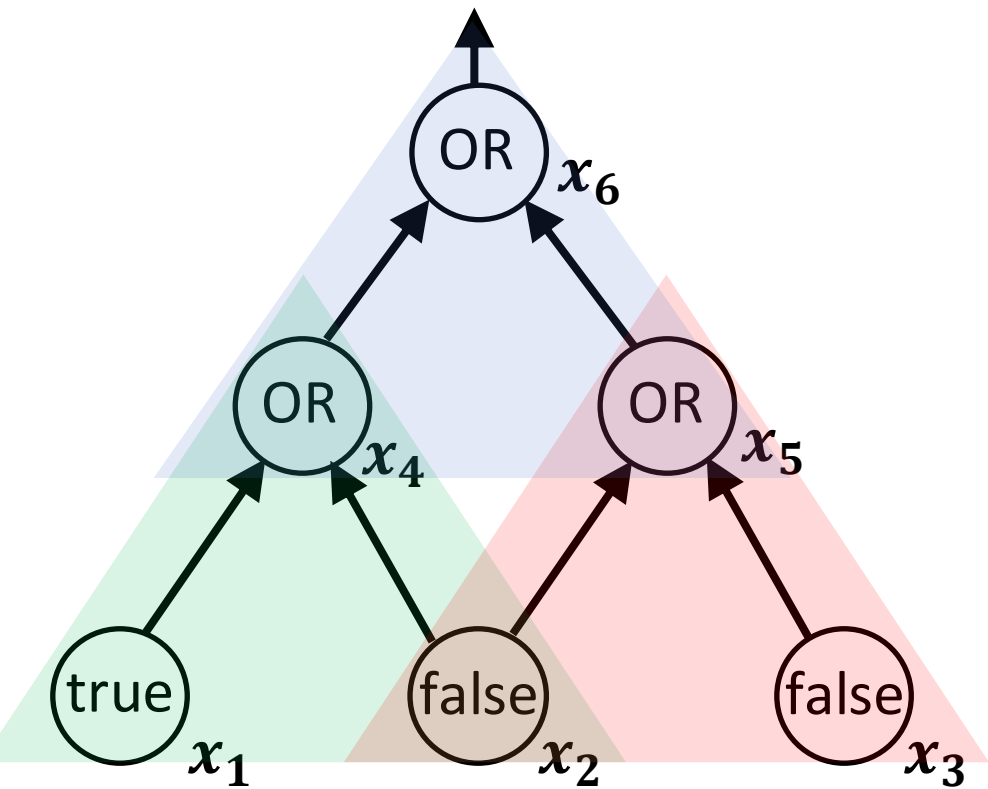
Will answer be an integer?

Yes! Input are integers and constraints keep them integers.





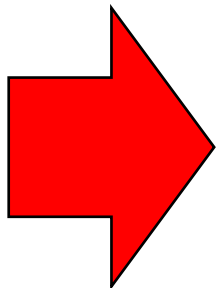
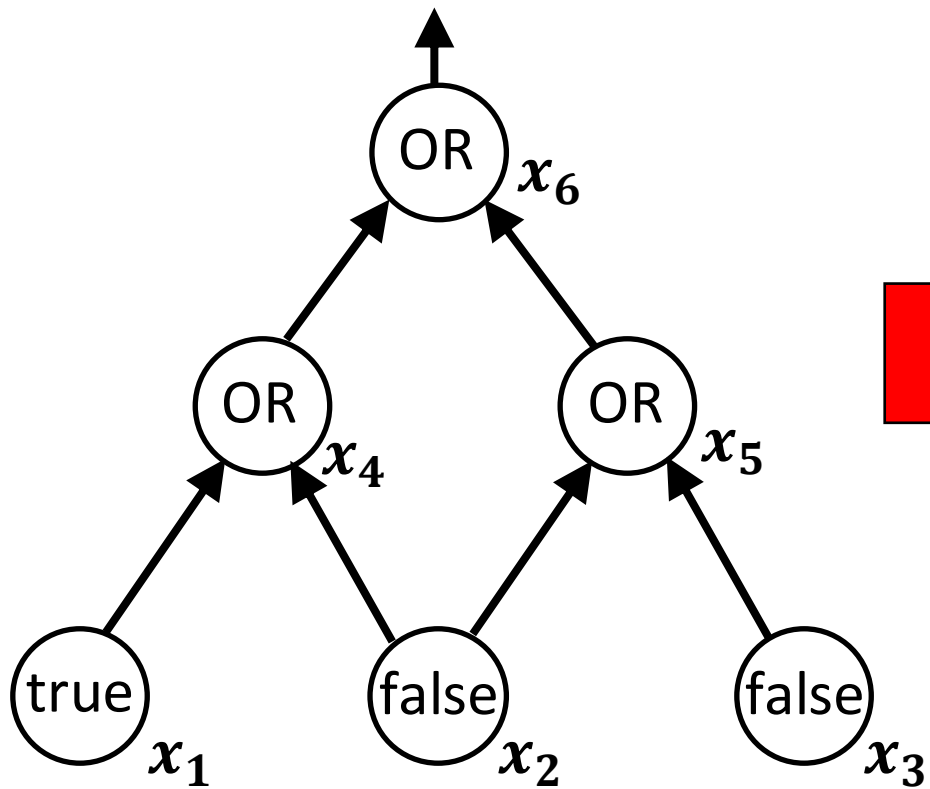
Will answer be an integer?



$A =$

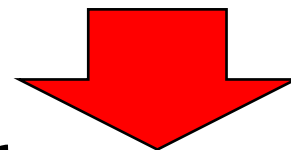
	x_1	x_2	x_3	x_4	x_5	x_6
c_1	1	0	0	-1	0	0
c_2	0	1	0	-1	0	0
c_3	-1	-1	0	1	0	0
c_4	0	1	0	0	-1	0
c_5	0	0	1	0	-1	0
c_6	0	-1	-1	0	1	0
c_7	0	0	0	1	0	-1
c_8	0	0	0	0	1	-1
c_9	0	0	0	-1	-1	1

Will answer be an integer?

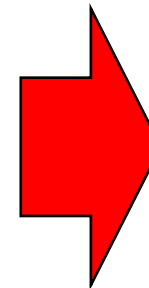


$A =$

	x_1	x_2	x_3	x_4	x_5	x_6
c_1	1	0	0	-1	0	0
c_2	0	1	0	-1	0	0
c_3	-1	-1	0	1	0	0
c_4	0	1	0	0	-1	0
c_5	0	0	1	0	-1	0
c_6	0	-1	-1	0	1	0
c_7	0	0	0	1	0	-1
c_8	0	0	0	0	1	-1
c_9	0	0	0	-1	-1	1



1	-1	0
1	0	-1
0	-1	-1



det = -2

Will answer be an integer?

NP

P { P is the set of problems that are solvable by an algorithm whose running time is polynomial time.

NP

P { P is the set of problems that are solvable by an algorithm whose running time is polynomial time.

NP { Set of problems that are verifiable in polynomial time.

SUBSET – SUM

Claim: $SUBSET – SUM = \{\langle S, t \rangle : S = \{x_1, \dots, x_n\}, \text{ and there exists some } \{y_1, \dots, y_m\} \subseteq S \text{ such that } \sum y_i = t\} \in NP.$

Example:

$\langle \{4, 11, 16, 21, 27\}, 25 \rangle \in SUBSET – SUM$ since $4 + 21 = 25$

$\langle \{1, 2, 3\}, 10 \rangle \notin SUBSET – SUM$, since no subsets sum to 10.

SUBSET – SUM

Claim: $SUBSET – SUM = \{\langle S, t \rangle : S = \{x_1, \dots, x_n\}, \text{ and there exists some } \{y_1, \dots, y_m\} \subseteq S \text{ such that } \sum y_i = t\} \in NP.$

Proof:

Solver: Is $\langle S, t \rangle \in SUBSET – SUM$?

Verifier: Is $\langle S, t \rangle \in SUBSET – SUM$, given a candidate solution?

Set of numbers.



SUBSET – SUM

Claim: $SUBSET – SUM = \{\langle S, t \rangle : S = \{x_1, \dots, x_n\}, \text{ and there exists some } \{y_1, \dots, y_m\} \subseteq S \text{ such that } \sum y_i = t\} \in NP$.

Proof: Build a polynomial time verifier.

SUBSET – SUM

Claim: $SUBSET – SUM = \{\langle S, t \rangle : S = \{x_1, \dots, x_n\}, \text{ and there exists some } \{y_1, \dots, y_m\} \subseteq S \text{ such that } \sum y_i = t\} \in NP$.

Proof: Build a polynomial time verifier.

On input $\langle \langle S, t \rangle, C \rangle$, where C is a collection of numbers

1. Test if every element from C is in S .
2. Test if the sum of C is t .
3. If both pass, accept. Otherwise, reject.

SUBSET – SUM

Claim: $SUBSET – SUM = \{\langle S, t \rangle : S = \{x_1, \dots, x_n\}, \text{ and there exists some } \{y_1, \dots, y_m\} \subseteq S \text{ such that } \sum y_i = t\} \in NP$.

Proof: Build a polynomial time verifier.

On input $\langle \langle S, t \rangle, C \rangle$, where C is a collection of numbers

1. Test if every element from C is in S .
2. Test if the sum of C is t .
3. If both pass, accept. Otherwise, reject.

Verified in $O(n)$ time, therefore $SUBSET – SUM \in NP$.

P vs NP

P { P is the set of problems that are solvable by an algorithm whose running time is polynomial time.

NP { Set of problems that are verifiable in polynomial time.

P vs NP

P { P is the set of problems that are solvable by an algorithm whose running time is polynomial time.

NP { Set of problems that are verifiable in polynomial time.

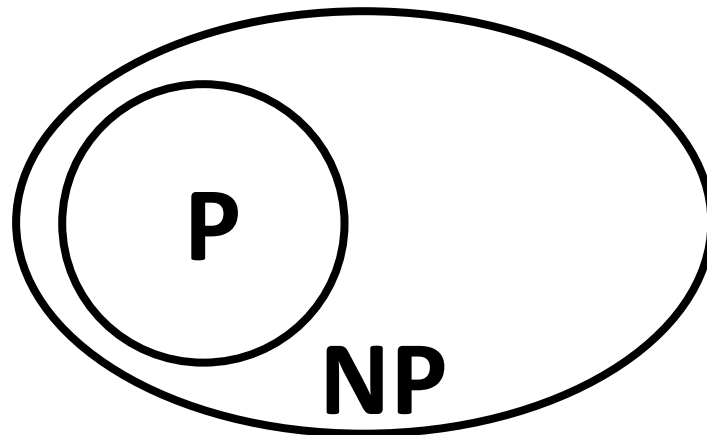
How does P relate to NP ?

P vs NP

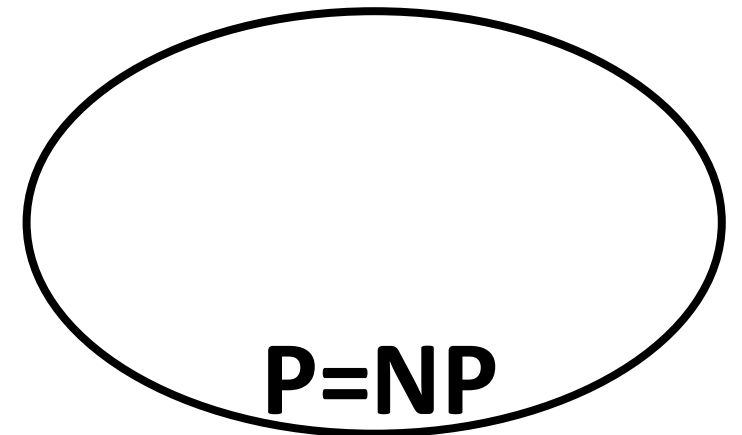
P { P is the set of problems that are solvable by an algorithm whose running time is polynomial time.

NP { Set of problems that are verifiable in polynomial time.

How does P relate to NP ?



Or



NP
Complete { Set of problems in NP whose algorithms
can solve any other problem in NP with
polynomial extra time.

